

Euler Characteristics as Invariants of Iwasawa Modules*

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Introduction

Let G be a pro- p , p -adic, Lie group with no element of order p and let $\Lambda(G)$ denote the Iwasawa algebra of G , defined in the usual way by

$$\Lambda(G) = \varprojlim \mathbb{Z}_p[G/H], \quad (1)$$

where H runs over the open, normal subgroups of G , and the inverse limit is taken with respect to the canonical projection maps. Then many situations arise in which one is interested in determining information about the structure of modules which are finitely generated over $\Lambda(G)$, such as the rank (defined in (2) below). In this paper we describe a number of invariants associated to a finitely generated $\Lambda(G)$ module, M , and calculated via an Euler characteristic formula. For example, we give a simple formula for the rank of M in terms of an Euler Characteristics formula. These ideas are well known to Algebraists, see for example the book by K. Brown, [2], chapter IX in particular, but do not appear to have been exploited to their full potential in Iwasawa Theory yet. This formula gives the natural generalisation of the strongest form of Nakayama's lemma (for the case of G isomorphic to \mathbb{Z}_p) to other pro- p , p -adic, Lie groups. Thus the first subsection of this paper should be seen as a continuation of the earlier note [1] which explained situations where that result can fail to generalise directly.

We then apply these ideas to modules finitely generated over the \mathbb{F}_p -linear completed group algebra, and consider an invariant of Iwasawa modules which gives the classical Iwasawa μ -invariant in the case $G \cong \mathbb{Z}_p$, where the idea of expressing this invariant in terms of an Euler characteristic is well known.

If one instead starts with the Euler characteristic formula as the definition of a 'homological $\Lambda(G)$ -rank' then we can extend it to some situations where the naïve definition of $\Lambda(G)$ -rank is not appropriate. For example, we can consider removing the restriction that G is a pro- p group. In the second section we give some discussion of the information this can tell us about a $\Lambda(G)$ -module. In particular, in the classical case of $G \cong \mathbb{Z}_p^\times$ this relates to the decomposition of a finitely generated

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$\Lambda(G)$ -module into its eigenspaces for the action of the subgroup $\mathbb{Z}/p\mathbb{Z}^\times$ of \mathbb{Z}_p^\times , a technique which is not available to us for an arbitrary p -adic, Lie group.

We also consider Euler characteristics of modules which are not actually finitely generated. In this way we can shed some light on a question raised in [6] concerning a generalisation of Iwasawa's classical λ -invariant, which has proved important in the theory for \mathbb{Z}_p -extensions of fields. We consider in detail this invariant and the generalisation of the μ -invariant for the dual of the Selmer group of an elliptic curve over a certain naturally defined p -adic, Lie extension, and analyse the behaviour of this second invariant under isogeny subject to some restrictions.

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1 Ranks of Iwasawa Modules

1.1 Λ -modules

Recall that G denotes a p -adic, Lie group and $\Lambda(G)$ denotes the Iwasawa algebra of G , defined in (1). Let M be any finitely generated $\Lambda(G)$ -module. Then we are interested in the $\Lambda(G)$ -rank of M . For the most natural definition of this we need $\Lambda(G)$ to contain no non-trivial zero divisors. It is known that this is the case if G contains no non-trivial element of finite order (see [17]) and we assume this for the remainder of this section. In particular, this means G is actually pro- p . Then $\Lambda(G)$ is both right and left Noetherian and thus admits a skew field of fractions which we denote by $K(G)$ (see [10], chapter 9.)

Definition Under the assumption that G contains no non-trivial element of finite order, the $\Lambda(G)$ -rank of a finitely generated $\Lambda(G)$ -module, M , is defined by

$$\text{rank}_{\Lambda(G)}(M) = \dim_{K(G)}(K(G) \otimes_{\Lambda(G)} M). \quad (2)$$

When this number is zero then we say M is $\Lambda(G)$ -torsion. If this is the case, then every element of M has a non-trivial annihilator in $\Lambda(G)$, though it should be emphasised that in the non-Abelian setting this is not sufficient to ensure the existence of a non-trivial element in $\text{ann}_{\Lambda(G)}(M)$.

We will replace this later (equation (46)) with a less intuitive, homological definition

of $\Lambda(G)$ –rank which turns out to be better suited for generalising. In particular, it provides further insight into a question raised in [6] which concerns modules that are not necessarily finitely generated. Theorem 1.1 of this section then states that the two definitions coincide for finitely generated $\Lambda(G)$ –modules.

We start by recalling the natural duality between cohomology and homology. If M is a $\Lambda(G)$ –module then we denote the Pontrjagin dual of M by

$$\widehat{M} = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p), \quad (3)$$

which is naturally also a $\Lambda(G)$ –module. Let \mathcal{C} denote the category of finitely generated $\Lambda(G)$ –modules, \mathcal{D} the category of discrete, p –primary $\Lambda(G)$ –modules (with continuous $\Lambda(G)$ action) which are cofinitely generated. We recall here that a $\Lambda(G)$ –module, D , is said to be cofinitely generated if \widehat{D} is finitely generated. Then $\widehat{}$ defines functors $\mathcal{D} \longrightarrow \mathcal{C}$ and $\mathcal{C} \longrightarrow \mathcal{D}$ (which for convenience we also denote by $\widehat{}$) cf. 2.3 of [3], which discusses this in greater generality. Taking the G –invariants, D^G , of a $\Lambda(G)$ –module D in \mathcal{D} gives a left exact functor from \mathcal{D} to the category of cofinitely generated \mathbb{Z}_p –modules. Similarly, taking the G –coinvariants (the maximal quotient of M on which G acts trivially), M_G , of a $\Lambda(G)$ –module M in \mathcal{C} gives a right exact functor from \mathcal{C} to the category of finitely generated \mathbb{Z}_p –modules. Since there is a canonical isomorphism

$$\widehat{M_G} \cong \widehat{M}^G \quad (4)$$

for any M in \mathcal{C} , this induces a canonical isomorphism between the corresponding derived functors,

$$H^i(G, \widehat{M}) \cong H_i(\widehat{G}, \widehat{M}), \quad (5)$$

for any finitely generated $\Lambda(G)$ –module, M . For convenience we prove our formula for modules in the category \mathcal{C} , and use homology. Many applications in Number Theory, however, are phrased in terms of the G –cohomology of modules in \mathcal{D} and thus we state our theorem in this language also.

Our first result is the following:

Theorem 1.1 *Assume G is a pro- p , p -adic, Lie group which contains no element of order p . Let M be a finitely generated $\Lambda(G)$ –module. Then the $\Lambda(G)$ –rank of M is given by the following ‘Euler Characteristic’ formula:*

$$\text{rank}_{\Lambda(G)}(M) = \sum_{i \geq 0} (-1)^i \text{rank}_{\mathbb{Z}_p}(H_i(G, M)). \quad (6)$$

Equivalently, for any D in the category \mathcal{D} of cofinitely generated, discrete $\Lambda(G)$ -modules,

$$\text{rank}_{\Lambda(G)}(\widehat{D}) = \sum_{i \geq 0} (-1)^i \text{corank}_{\mathbb{Z}_p}(H^i(G, D)). \quad (7)$$

In particular, both these sums are well-defined and finite.

We observe that equality (7) follows immediately from equality (6) because of the duality in (5).

Definition Let G be a pro- p , p -adic, Lie group which contains no element of order p . For any finitely generated $\Lambda(G)$ -module, M , we define $\chi_*(G, M)$, respectively $\chi^*(G, M)$, to be the integer which is equal to the sum which occurs on the right hand side of equality (6), respectively (7).

The following corollaries are immediate.

Corollary 1.2 *Under the hypotheses of Theorem 1.1, if the $H_i(G, M)$ are finite for all $i \geq 0$ then M is $\Lambda(G)$ -torsion.*

In fact, since the $\Lambda(G)$ -rank of any M in \mathcal{C} cannot be negative, it is sufficient to know the finiteness of $H_i(G, M)$ for i even, only. As remarked above, it is interesting to compare this with the stronger results previously known in the classical case of $G \cong \mathbb{Z}_p$, usually denoted by Γ . In this situation the following result is well known.

Lemma 1.3 *Suppose Γ is isomorphic to \mathbb{Z}_p , and M is a compact $\Lambda(\Gamma)$ -module. If M_Γ is a finite group then M is $\Lambda(\Gamma)$ -torsion.*

This extremely useful condition for a finitely generated $\Lambda(\Gamma)$ -module to be torsion was first pointed out by Iwasawa and follows easily from the general structure theorem for finitely generated $\Lambda(\Gamma)$ -modules. Note that it is immediate from this structure theorem that if M_Γ is finite then so also is M^Γ . However, this now follows directly from Theorem 1.1 above, without the crutch of the structure theorem. Indeed, Lemma 1.3 follows directly from Corollary 1.2, and the comment which follows it, together with the equality $H_0(\Gamma, M) = M_\Gamma$ and the fact that the cohomological dimension at p of \mathbb{Z}_p is equal to 1. Then, since $H_1(\Gamma, M) = M^\Gamma$, equality (6) implies that M^Γ is finite. In [1] it was shown that Lemma 1.3 generalises to uniform, soluble, p -adic, Lie groups. However, as was explained there, Lemma 1.3 fails for arbitrary insoluble groups, even under the assumption of being uniform. Thus

Theorem 1.1 seems to be the natural generalisation for arbitrary torsion free, pro- p , p -adic, Lie groups. See [9] for a further discussion of the technical condition 'uniform'. We remark that \mathbb{Z}_p and also any congruence kernel of $\mathrm{GL}_n(\mathbb{Z}_p)$ are uniform groups. As we will need the concept again later we define it here.

Definition A profinite group is *uniform* if it is topologically finitely generated on d generators, and there exists a filtration by subgroups

$$G = G_0 \supset G_1 \supset \cdots \quad (8)$$

such that each G_i is normal in G_{i+1} , and $G_i/G_{i+1} \cong (\mathbb{Z}/p\mathbb{Z})^d$. In particular, a uniform p -adic analytic group is always pro- p .

Remark A fundamental fact is that a profinite group is p -adic analytic if and only if it contains an open, normal subgroup which is uniform [9].

Corollary 1.4 *Assume the hypotheses of Theorem 1.1. If M is finitely generated as a \mathbb{Z}_p -module then*

$$\sum_{i \geq 0} (-1)^i \mathrm{rank}_{\mathbb{Z}_p}(H_i(G, M)) = 0. \quad (9)$$

The only comment necessary here is to note that since we are assuming G contains no element of order p , if it is not trivial (in which case Corollary 1.4 is vacuous) then it must be infinite. Thus $\Lambda(G)$ is not finitely generated as a \mathbb{Z}_p -module and so if M is finitely generated as a \mathbb{Z}_p -module then it must be $\Lambda(G)$ -torsion.

The main application of these results in Number Theory is in Iwasawa Theory, where one is interested in studying the behaviour of arithmetic objects over infinite towers of fields. The typical situation is the following. Let F be a finite extension of \mathbb{Q} and F_∞ a p -adic, Lie extension of F , that is a Galois field extension with $G = \mathrm{Gal}(F_\infty/F)$ isomorphic to a p -adic, Lie group. Then certain modules in \mathcal{D} are of great arithmetic interest. For example the cohomology groups

$$H^i(F^S/F_\infty, A), \quad (10)$$

where F^S is the maximal extension of F unramified outside a chosen finite set of primes of F , including all the primes which divide p or which ramify in the extension F_∞/F (assumed to be a finite set), and A is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ as a \mathbb{Z}_p -module, endowed with a continuous action of $\mathrm{Gal}(F^S/F)$.

The application of Theorem 1.1 to studying these modules, among other examples, will be described in [13]. Unfortunately, it is often hard to calculate higher homology and cohomology groups ($i \geq 2$) and so Theorem 1.1 provides limited help, for example, with calculating the $\Lambda(G)$ -rank of the p^∞ -Selmer group of an elliptic curve, as discussed in [6]. We will also discuss this situation in more detail later.

It is necessary to recall some general definitions and properties of p -adic, Lie groups.

Definition Recall that the p -cohomological dimension of G , denoted throughout by $\text{cd}_p(G)$, is defined as the minimum number such that $H^i(G, D)$ vanishes for all discrete, p -primary, $\Lambda(G)$ -modules, D , whenever $i > \text{cd}_p(G)$. It follows from (5) that for all finitely generated $\Lambda(G)$ -modules, M , one also has $H_i(G, M) = 0$ for all $i > \text{cd}_p(G)$.

If M is in \mathcal{C} then the $\Lambda(G)$ -homological dimension of M is the minimum integer, n , such that M has a resolution of length n by projective $\Lambda(G)$ -modules. (The $\Lambda(G)$ -homological dimension is said to be infinite if no finite resolution exists.) We define the global dimension, $\text{gldim}(\Lambda(G))$, of $\Lambda(G)$ to be the supremum of the homological dimension of M , as M ranges over \mathcal{C} .

Properties of p -adic, Lie groups *i) Any p -adic, Lie group containing no element of order p (but which is not necessarily pro- p) has finite p -cohomological dimension, equal to its dimension as a p -adic Lie group, [15] and [21].*

ii) It is shown in [3] that under the hypotheses of i) $\Lambda(G)$ has finite global dimension, given by

$$\text{gldim}(\Lambda(G)) = 1 + \text{cd}_p(G). \quad (11)$$

iii) For G pro- p , but not necessarily torsion free, $\Lambda(G)$ is a local ring, with unique maximal ideal given by the kernel of the augmentation map, $\Lambda(G) \rightarrow \mathbb{Z}/p\mathbb{Z}$, [15].

Proof of Theorem 1.1.

We fix G throughout the proof of Theorem 1.1 and write Λ for $\Lambda(G)$. We also fix a finitely generated Λ -module, M . The finiteness of the global dimension of Λ means that M admits a projective resolution of finite length (at most equal to $\text{gldim}(\Lambda)$, by definition.) Because Λ is a local ring, the only projective Λ -modules are the free modules. Thus M admits a resolution of finite length, d say, by free modules. Fix

such a resolution:

$$0 \rightarrow \Lambda^{n_d} \rightarrow \cdots \rightarrow \Lambda^{n_1} \rightarrow \Lambda^{n_0} \rightarrow M \rightarrow 0. \quad (12)$$

It follows that the groups $H_i(G, M)$ can be computed as the homology of a complex of finite length of the form

$$\mathbb{Z}_p^{n_d} \longrightarrow \cdots \longrightarrow \mathbb{Z}_p^{n_1} \longrightarrow \mathbb{Z}_p^{n_0} \quad (13)$$

immediately implying that each $H_i(G, M)$ is finitely generated over \mathbb{Z}_p and that

$$\chi_*(G, M) = \sum_{i \geq 0} (-1)^i n_i. \quad (14)$$

On the other hand, since $K(G)$ is a flat Λ -module, the alternating sum of Λ -ranks adds to zero along exact sequences. Thus it follows from (12) that

$$\text{rank}_\Lambda(M) = n_0 - n_1 + n_2 \cdots + (-1)^d n_d, \quad (15)$$

completing the proof of Theorem 1.1. □

The following simple Corollary will be needed later.

Corollary 1.5 *If G_0 is an open normal subgroup in G then the rank of a finitely generated $\Lambda(G)$ -module over $\Lambda(G_0)$ equals $|G : G_0|$ times its rank over $\Lambda(G)$. In particular, a $\Lambda(G)$ -module is $\Lambda(G)$ -torsion if and only if it is $\Lambda(G_0)$ -torsion.*

Proof This is immediate from the fact that $\Lambda(G)$ is free over $\Lambda(G_0)$, of rank $|G : G_0|$, and so a free resolution of M as a $\Lambda(G)$ -module also gives a free resolution of M as a $\Lambda(G_0)$ -module, with alternating sum of $\Lambda(G_0)$ -ranks equal to $|G : G_0|$ times the alternating sum of $\Lambda(G)$ -ranks. □

Example As a simple illustration of the application of Theorem 1.1 we consider the smallest concrete example discussed in [1] of a p -adic, Lie group for which the assertion of Lemma 1.3 can fail.

Assume $p \geq 5$ and let

$$H = \{A \in \text{SL}_2(\mathbb{Z}_p) \mid A \equiv I, \text{ mod } p\}. \quad (16)$$

Then H is not soluble (that is, its derived series does not terminate after a finite number of terms) and it can be shown that

$$\overline{[H, H]} = \{A \in \mathrm{SL}_2(\mathbb{Z}_p) \mid A \equiv I, \text{ mod } p^2\}, \quad (17)$$

where $\overline{[H, H]}$ is the group generated *topologically* by $[H, H]$. Now H is a pro- p , p -adic, Lie group of dimension 3. Since $p \geq 5$ it contains no element of order p , and so Theorem 1.1 applies. Let $I(H)$ denote the augmentation ideal of $\Lambda(H)$, that is the kernel of the canonical augmentation map, ε , to \mathbb{Z}_p

$$0 \longrightarrow I(H) \longrightarrow \Lambda(H) \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0. \quad (18)$$

Since the alternating sum of $\Lambda(H)$ -ranks adds along exact sequences it is clear that

$$\mathrm{rank}_{\Lambda(H)}(I(H)) = 1. \quad (19)$$

We now calculate the homology groups $H_i(H, I(H))$, in order to illustrate that the non-zero contribution to the sum in the formula (6) can come entirely from the terms with $i \neq 0$. We start by taking homology of (18). This gives

$$\begin{aligned} H_0(H, I(H)) &\cong I(H)/(I(H))^2 \\ H_i(H, I(H)) &\cong H_{i+1}(H, \mathbb{Z}_p), & i \geq 1 \\ &= 0, & i \geq 3, \end{aligned} \quad (20)$$

since $cd_p(H) = 3$. It is easy to see that as an Abelian group

$$I(H)/(I(H))^2 \cong H/\overline{[H, H]}, \quad (21)$$

which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$ by (16) and (17) and thus in particular is finite. To calculate $H_i(H, \mathbb{Z}_p)$ for $i = 2, 3$ we use the fact, shown in [15] Theorem 2.5.8, that any pro- p , p -adic, Lie group is a Poincaré group of dimension equal to its dimension as a p -adic manifold. The definition of a Poincaré group is somewhat technical and we simply recall the following basic property:

If G is a Poincaré group of dimension n then the cup product induces a perfect pairing

$$H^i(G, A) \times H^{n-i}(G, \tilde{A}) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \quad (22)$$

for A any finite G -module endowed with the discrete topology. Here \tilde{A} denotes $\mathrm{Hom}_{cts}(A, I)$, endowed with the natural action of G , where I is the dualising module of G . We shall not define dualising modules in general. In fact, H is what is known as

an *orientable* Poincaré group, in other words its dualising module is simply $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial action of H . See [15] and [22] for a more detailed discussion of the definition and properties of Poincaré groups.

From the corollary to Proposition 2.2 of [23],

$$H_{cts}^i(H, \mathbb{Z}_p) = \varprojlim H^i(H, \mathbb{Z}/p^n\mathbb{Z}), \quad (23)$$

where the inverse limit is taken with respect to the canonical maps. Here $H_{cts}^i(H, \mathbb{Z}_p)$ denotes cohomology calculated with continuous cochains and where \mathbb{Z}_p is endowed with the usual profinite topology. It follows from this, the isomorphism in (5) and the pairing (22) above that

$$H_i(H, \mathbb{Z}_p) \cong H_{cts}^{3-i}(H, \mathbb{Z}_p). \quad (24)$$

Since $H_{cts}^1(H, \mathbb{Z}_p)$ is just the group of continuous group homomorphisms of H into \mathbb{Z}_p , it follows from (24) and the finiteness of $H^{ab} = H/\overline{[H, H]}$ that $H_2(H, \mathbb{Z}_p)$ vanishes and so $H_1(H, I(H))$ vanishes also. Whereas putting together (20) and (24) for i equal to 2, we see

$$H_2(H, I(H)) \cong H_3(H, \mathbb{Z}_p) \cong H_{cts}^0(H, \mathbb{Z}_p) \cong \mathbb{Z}_p. \quad (25)$$

So the only non-zero contribution to the formula (6) is from the term $H_2(H, I(H))$, of \mathbb{Z}_p -rank equal to 1.

1.2 Ω -modules

We include here a short subsection concerning \mathbb{F}_p -linear, completed group algebras. This is interesting with regard to our attempts to understand more generally the possible structures of modules over Iwasawa algebras. Otmar Venjakob has independently considered the structure of modules over the \mathbb{F}_p -linear, completed group algebra in some detail in his Heidelberg PhD. thesis, [24].

We use the notation

$$\Omega(G) = \varprojlim \mathbb{F}_p[G/H], \quad (26)$$

for the \mathbb{F}_p -linear, completed group algebra, and continue with the assumption that G contains no non-trivial element of finite order. Clearly $\Omega(G) = \Lambda(G)/p\Lambda(G)$ and the following properties of $\Omega(G)$ hold, as for $\Lambda(G)$.

Properties *i) The global dimension of $\Omega(G)$ is finite, this time equal to the cohomological dimension of G at p , since \mathbb{F}_p is a field.*

ii) $\Omega(G)$ is a right and left Noetherian, local ring.

iii) As remarked just prior to the statement of Corollary 1.4, G contains an open normal subgroup, G_0 , which is uniform. The associated group algebra $\Omega(G_0)$ has no non-trivial zero divisors (see the second edition of [9]), thus admits a skew field of fractions, which we denote by $k(G_0)$.

Unfortunately, the proof in [17] that $\Lambda(G)$ contains no non-trivial zero divisors when G is torsion free fails to generalise to characteristic p , and it appears to be unknown whether $\Omega(G)$ is torsion free without the restriction that G be uniform.

Definition Similarly to the case for $\Lambda(G)$ -modules, the $\Omega(G_0)$ -rank of a finitely generated $\Omega(G)$ -module, M , is given by $\dim_{k(G_0)}(k(G_0) \otimes_{\Omega(G_0)} M)$. Since $\Omega(G)$ is finitely generated over $\Omega(G_0)$, M is also finitely generated over $\Omega(G_0)$. Thus we extend this definition and define the $\Omega(G)$ -rank as

$$\text{rank}_{\Omega(G)}(M) = \frac{\text{rank}_{\Omega(G_0)}(M)}{|G : G_0|}. \quad (27)$$

We will show below that this is integral and independent of the choice of G_0 .

Throughout the remainder we will use the notation

$$\chi(G, M) = \prod_{i \geq 0} (\#H_i(G, M))^{(-1)^i} \quad (28)$$

for any $\Lambda(G)$ -module, M , for which the terms in this product are finite and almost all equal to one.

Then the arguments of section 1.1 all follow through also for finitely generated $\Omega(G)$ -modules. In particular, one has the following result.

Proposition 1.6 *Assume G is a pro- p , p -adic, Lie group which contains no element of order p . Let M be a finitely generated $\Omega(G)$ -module. Then the $\Omega(G)$ -rank of M is given by the following, finite, sum*

$$\text{rank}_{\Omega(G)}(M) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{F}_p}(H_i(G, M)). \quad (29)$$

By the definition above, this is equal to $\text{ord}_p(\chi(G, M))$, where $\text{ord}_p(\alpha)$ denotes the maximum power of p which divides α . In particular, for any finitely generated $\Omega(G)$ -module, M , it follows that $\chi(G, M)$ is always well-defined and that the definition (27)

of the $\Lambda(G)$ –rank is both independent of the choice of subgroup, G_0 , used to define it and integral.

Proof This follows by almost exactly the same arguments used above to prove Theorem 1.1. Since $\Omega(G)$ is a local ring and has finite global dimension, we may again take a finite, free resolution of M , this time as an $\Omega(G)$ –module,

$$0 \longrightarrow \Omega(G)^{n_d} \longrightarrow \cdots \longrightarrow \Omega(G)^{n_0} \longrightarrow M \longrightarrow 0. \quad (30)$$

The only extra point needed is that since $\Omega(G)$ is not a projective $\Lambda(G)$ –module, we must first calculate $\chi(G, \Omega(G)^n) = \chi(G, \Omega(G))^n$. Consider the sequence

$$0 \longrightarrow \Lambda(G) \xrightarrow{\times p} \Lambda(G) \longrightarrow \Omega(G) \longrightarrow 0. \quad (31)$$

Taking G –homology, and recalling $H_i(G, \Lambda(G)) = 0$ for $i \geq 1$, we obtain

$$0 \longrightarrow H_1(G, \Omega(G)) \longrightarrow \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p \longrightarrow H_0(G, \Omega(G)) \longrightarrow 0 \quad (32)$$

and $H_i(G, \Omega(G))$ vanishes for $i \geq 2$. Since $H_1(G, \Omega(G))$ is both annihilated by p and contained in \mathbb{Z}_p , it must be zero also, while $H_0(G, \Omega(G))$ is simply \mathbb{F}_p . Thus $\chi(G, \Omega(G)^n)$ equals n and similarly $\chi(G_0, \Omega(G)^n)$ equals $|G : G_0|n$. It follows that we can calculate the $H_i(G_0, M)$ by working in the category of finitely generated $\Omega(G_0)$ –modules and using the projective resolution (30). The remainder of the proof of Proposition 1.6 now follows exactly as before for Theorem 1.1, in particular $\Omega(G_0)$ is again flat as an $\Omega(G_0)$ –module and so the $\Omega(G_0)$ –rank of M is equal to $(n_0 - n_1 + \cdots \pm n_d) |G : G_0|$, from which the proposition is clear.

□

Corollaries 1.2 and 1.4 also have equivalent formulations for finitely generated $\Omega(G)$ –modules. In fact the analogue of Corollary 1.4 is already contained in the exercise at the end of §I.4 of [22] and is valid for more general groups, G . We will therefore omit reformulating statements or proofs.

Suppose now that M is a finitely generated $\Lambda(G)$ –module. Let $M(p)$ denote the subset of M consisting of the elements of M which are annihilated by some power of p (clearly a $\Lambda(G)$ –submodule as p is in the centre of $\Lambda(G)$.) Since $\Lambda(G)$ is Noetherian, $M(p)$ is again finitely generated, and thus there exists some integer $r \geq 0$ such that p^r annihilates $M(p)$. We propose the following invariant of $\Lambda(G)$ –modules.

Definition If M is a finitely generated $\Lambda(G)$ –module then we define

$$\mu(M) = \sum_{i \geq 0} \text{rank}_{\Omega(G)} \left(p^i(M(p)) / p^{i+1} \right) \quad (33)$$

Taken in conjunction with the discussion preceding this definition, the fact that each sub-quotient $\left(p^i(M(p))/p^{i+1}\right)$ is a finitely generated $\Omega(G)$ -module implies that this sum is finite. We have used the notation $\mu(M)$ since in the case $G \cong \mathbb{Z}_p$ this gives the classical μ -invariant of Iwasawa Theory for Γ -modules. This is a particularly convenient definition to work with.

Corollary 1.7 *Assume G contains no non-trivial element of finite order. If M is a finitely generated $\Lambda(G)$ -module then*

$$\mu(M) = \text{ord}_p(\chi(G, M(p))). \quad (34)$$

In particular, $\chi(G, M(p))$ is always finite.

Unfortunately, there exist finitely generated $\Lambda(G)$ -modules, M , for which $\chi(G, M)$ is finite and non-zero and yet $\mu(M)$ is equal to zero, and so we $\mu(M)$ cannot be defined solely in terms of $\chi(G, M)$. Indeed, as has been shown in [8], there actually exist modules T which are finitely generated and free over \mathbb{Z}_p and for which $\text{ord}_p(\chi(G, T))$ is strictly positive, or even strictly negative. These examples are obtained by twisting by roots of unity the Tate module of Tate curves defined over a finite extension of \mathbb{Q}_p . See [8], Corollary 5.3, for further details. In particular, $\chi(G, M)$ need not be integral if M is not p -torsion, even for rather small $\Lambda(G)$ -modules, and allowing G to be uniform.

Proof We consider the short exact sequences

$$0 \longrightarrow p^{i+1}(M(p)) \longrightarrow p^i(M(p)) \longrightarrow p^i(M(p))/p^{i+1} \longrightarrow 0 \quad (35)$$

obtained from the filtration

$$M(p) \supset p(M(p)) \supset \cdots \supset p^r(M(p)) = 0 \quad (36)$$

for r sufficiently large. The modules $p^i(M(p))/p^{i+1}$ are finitely generated $\Omega(G)$ -modules for all i and so, by Proposition 1.6

$$\text{ord}_p \chi(G, p^i(M(p))/p^{i+1}) = \text{rank}_{\Omega(G)}(p^i(M(p))/p^{i+1}). \quad (37)$$

In particular, these numbers are all defined. Since $p^r(M(p))$ vanishes, the Euler characteristic $\chi(G, p^{r-1}(M(p)))$ is finite by Proposition 1.6. It then follows inductively from the long exact sequence in cohomology that $\chi(G, p^i(M(p)))$ is finite for

all i , equal to the product of $\chi(G, p^{i+1}(M(p)))$ and $\chi(G, p^i(M(p))/p^{i+1})$. Multiplying these together, we obtain the formula (34) of the Corollary.

□

Although this invariant is clearly not additive along exact sequences (consider, for example, the sequence $0 \rightarrow p\Lambda(G) \rightarrow \Lambda(G) \rightarrow \Omega(G) \rightarrow 0$) it does behave well when restricted to torsion $\Lambda(G)$ -modules.

Proposition 1.8 *Assume G contains no non-trivial element of finite order. In a short exact sequence of finitely generated $\Lambda(G)$ -modules,*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0, \quad (38)$$

one has $\mu(B) \leq \mu(A) + \mu(C)$, with equality if B , and hence also A and C , is $\Lambda(G)$ -torsion.

Proof We start with the observation that if A, B and C are p -torsion then the lemma follows from Corollary 1.7, since all Euler characteristics are defined and alternating products of Euler characteristics along exact sequences multiply together to give 1. In general we have $0 \rightarrow A(p) \rightarrow B(p) \rightarrow C(p)$, which, together with this remark gives the inequality. However, exactness can fail on the right. Let X denote the image of $B(p)$ in $C(p)$. It follows from the first remark that $\mu(B) = \mu(A) + \mu(X)$. We show that if B is $\Lambda(G)$ -torsion, then $\mu(C(p)/X) = 0$ and thus $\mu(C) = \mu(X)$. The surjection $B \twoheadrightarrow C$ induces a surjection $B/B(p) \twoheadrightarrow C/X$. By assumption, B is $\Lambda(G)$ -torsion, and thus $B/B(p)$ is also $\Lambda(G)$ -torsion. By Corollary 1.5 it is also $\Lambda(G_0)$ -torsion for any choice of uniform, open, normal subgroup of G . Let G_0 be any such subgroup. Then $B/B(p)$ is $\Lambda(G_0)$ -torsion and p -torsion free. Since p is in the centre of $\Lambda(G_0)$, it follows that for every element c of C/X there exists an element, λ , of $\Lambda(G_0)$ which is not contained in $p\Lambda(G_0)$ and which annihilates c . In particular, this is true for all elements of $C(p)/X$.

Lemma 1.9 *Let G_0 be a uniform, pro- p group. Suppose $Z = \Lambda(G_0)\alpha$ is a p -torsion $\Lambda(G_0)$ -module, generated by a single element, α , and such that there exists an element λ of $\Lambda(G_0)$, not contained in $p\Lambda(G_0)$, for which $\lambda\alpha = 0$. Then $\mu(Z) = 0$.*

Proof Since p is in the centre of $\Lambda(G_0)$ we may consider the modules $p^i Z/p^{i+1}$, equal to $\Lambda(G_0)p^i\alpha/p^{i+1}$, separately for each i and thus, without loss of generality, we may assume Z is annihilated by p , that is, Z is an $\Omega(G_0)$ -module generated

by a single element, α . Furthermore, this generator α has a non trivial annihilator in $\Omega(G_0)$ given by the image of λ under the canonical surjection of $\Lambda(G_0)$ onto $\Omega(G_0)$. But then it is immediate this means Z has rank zero as an $\Omega(G_0)$ -module, as required.

□

By the definition of $\Omega(G)$ -ranks, for any finitely generated $\Omega(G)$ -module, N , the μ -invariant, $\mu(N)$, calculated as an $\Omega(G)$ -module vanishes if and only if it vanishes when considering N as an $\Omega(G_0)$ -module. The module $C(p)/X$ is finitely generated over $\Lambda(G_0)$, as $\Lambda(G_0)$ is Noetherian and $\Lambda(G)$ is finitely generated over $\Lambda(G_0)$. Thus the proposition follows by induction on the number of generators of $C(p)/X$.

□

We now finish this section with a result which compares the $\Omega(G)$ -rank of M/p with the $\Lambda(G)$ -rank of M .

Corollary 1.10 *Assume G contains no non-trivial element of finite order. Let M be a finitely generated $\Lambda(G)$ -module. Then*

$$\text{rank}_{\Omega(G)}(M/pM) = \text{rank}_{\Omega(G)}(M[p]) + \text{rank}_{\Lambda(G)}(M), \quad (39)$$

where $M[p]$ denotes the submodule of M consisting of the set of elements, m , contained in M such that $pm = 0$.

Proof We begin by considering a finitely generated $\Lambda(G)$ -module, N , which is actually p -torsion. Since N is p -torsion, it follows from Corollary 1.7 that $\chi(G, N)$, $\chi(G, N[p])$ and $\chi(G, N/pN)$ are all finite, where $\chi(G, -)$ is the Euler characteristic in the category of $\Lambda(G)$ -modules as defined by (28). Upon consideration of the exact sequence :

$$0 \longrightarrow N[p] \longrightarrow N \longrightarrow N/pN \longrightarrow 0, \quad (40)$$

and since the alternating product of Euler characteristics along an exact sequence multiplies together to make 1, this shows that

$$\chi(G, N[p]) = \chi(G, N/pN). \quad (41)$$

However, both $N[p]$ and N/pN actually have exponent p and so, by Proposition 1.6, this shows that they have equal rank as $\Omega(G)$ -modules. Thus (39) holds for finitely generated p -torsion modules. In particular, it holds for $M(p)$.

Because $M/M(p)$ is p -torsion free, the sequence

$$0 \longrightarrow M(p)/pM(p) \longrightarrow M/pM \longrightarrow \frac{M/M(p)}{p(M/M(p))} \longrightarrow 0 \quad (42)$$

is exact, and so

$$\text{rank}_{\Omega(G)}(M/pM) = \text{rank}_{\Omega(G)}\left(\frac{M/M(p)}{p(M/M(p))}\right) + \text{rank}_{\Omega(G)}(M[p]). \quad (43)$$

Since the $\Lambda(G)$ -rank of $M/M(p)$ equals that of M , it only remains to show that if N is a p -torsion free $\Lambda(G)$ -module, then the $\Omega(G)$ -rank of N/pN equals the $\Lambda(G)$ -rank of N . But this follows immediately from Theorem 1.1, Proposition 1.6 and the long exact sequence in G -homology of

$$0 \longrightarrow N \xrightarrow{\times p} N \longrightarrow N/p \longrightarrow 0. \quad (44)$$

Indeed, the multiplication by p map on N induces multiplication by p on the homology groups, $H_i(G, N)$. Since on a finite Abelian group the kernel and cokernel of multiplication by p have the same order, we easily see that

$$\text{ord}_p(\chi(G, N/p)) = \chi_*(G, N). \quad (45)$$

By Theorem 1.1, the term on the right equals the $\Lambda(G)$ -rank of N , while by Proposition 1.6 the term on the left equals the $\Omega(G)$ -rank of N/pN .

□

2 Homological Λ -ranks

We have proven that under the conditions of Theorem 1.1 the right hand side of (6) is always finite. However, there are other situations where $\chi_*(G, M)$ can be finite and the naïve definition of rank is no longer suitable. Thus we propose the following definition.

Definition Let G be any p -adic, Lie group with no elements of order p , and $\Lambda(G)$ the Iwasawa algebra of G . Let M be any compact $\Lambda(G)$ -module. Then we say M has *finite homological rank* if the sum on the right of (6), $\chi_*(G, M)$, has finitely many non-zero, but finite, terms and is thus well-defined. In which case we define

$$\text{hmrk}_{\Lambda(G)}(M) = \chi_*(G, M). \quad (46)$$

So Theorem 1.1 states that, for G a pro- p , p -adic, Lie group, containing no element of order p ,

$$\text{hmrk}_{\Lambda(G)}(M) = \text{rank}_{\Lambda(G)}(M), \quad (47)$$

if M is finitely generated.

It follows from the long exact sequence in G -homology that

Lemma 2.1 *homological rank is additive along exact sequences. More precisely for any p -adic, Lie group, let*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \quad (48)$$

be an exact sequence of compact $\Lambda(G)$ -modules. Then if any two have finite homological rank, so does the third, in which case

$$\text{hmrnk}_{\Lambda(G)}(N) = \text{hmrnk}_{\Lambda(G)}(M) + \text{hmrnk}_{\Lambda(G)}(P). \quad (49)$$

We believe that this invariant will turn out to be very useful in a more general structure theory for $\Lambda(G)$ -modules. In particular, as evidence we consider the following two situations:

2.1 G fails to be pro- p

We can now consider p -adic, Lie groups, G , which are not pro- p . This is necessary for the groups which turn up 'in nature', for example $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ or $\text{Gal}(\mathbb{Q}(E_{p^\infty})/\mathbb{Q})$, where μ_{p^∞} is the set of all p^{th} -power roots of unity and $\mathbb{Q}(E_{p^\infty})$ is the field of definition of the p^{th} -power torsion points on an elliptic curve, E , defined over \mathbb{Q} . Both of these field extensions contain the extension $\mathbb{Q}(\mu_p)/\mathbb{Q}$ which has degree $p - 1$. The naïve definition of $\Lambda(G)$ -rank does not generalise. Indeed, if G is not a pro- p group then it will contain non-trivial elements of finite order and so $\Lambda(G)$ will contain zero divisors other than the zero element. Thus there is no skew field into which $\Lambda(G)$ can be embedded.

Lemma 2.2 *If M is a finitely generated $\Lambda(G)$ -module, for G a p -adic, Lie group containing no element of order p , then the homological rank of M is always finite.*

Proof From the remark just prior to the statement of Corollary 1.4, we know G contains a pro- p , normal subgroup of finite index, Thus the finiteness of the \mathbb{Z}_p -rank of $H_i(G, M)$ follows immediately from the case of a pro- p , p -adic, Lie group and the Hochschild-Serre spectral sequence for group cohomology.

□

It is not clear how to interpret this homological dimension, but an indication is the following.

Proposition 2.3 *Let G be a p -adic, Lie group containing no element of order p . Let G_0 be an open subgroup of G which is pro- p . Suppose M is a finitely generated $\Lambda(G)$ -module. Let $\text{res}_{G_0}^G M$ denote M considered as a $\Lambda(G_0)$ -module by restricting the action of $\Lambda(G)$. For any finitely generated $\Lambda(G_0)$ -module, N , let $\text{ind}_{G_0}^G N = N \otimes_{\Lambda(G_0)} \Lambda(G)$ be the algebraically induced $\Lambda(G)$ -module. Then*

$$\text{hmrk}_{\Lambda(G)}(\text{ind}_{G_0}^G(\text{res}_{G_0}^G M)) = \text{rank}_{\Lambda(G_0)}(M). \quad (50)$$

In particular, if G_0 is normal in G , with $\Delta = G/G_0$ then

$$\text{hmrk}_{\Lambda(G)}(M \bigotimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta]) = \text{rank}_{\Lambda(G_0)}(M) \quad (51)$$

where $\mathbb{Z}_p[\Delta]$ is a $\Lambda(G)$ -module via the canonical map $\Lambda(G) \rightarrow \mathbb{Z}_p[\Delta]$.

Proof This is an immediate consequence of Shapiro's Lemma together with Theorem 1.1. Recall that in general if G_0 is a closed subgroup of G and N is a discrete G_0 -module then $H^i(G_0, N)$ and $H^i(G, \text{ind}_{G_0}^G N)$ are naturally isomorphic (see for example [22], I.2.5.) Similarly for the homology of compact $\Lambda(G_0)$ -modules. □

We would like to thank R. Sujatha for pointing out how Proposition 2.3 follows easily from Shapiro's Lemma.

Our reason for including this elementary observation here is because in the case of G isomorphic to \mathbb{Z}_p^\times , with p at least 3, it is closely related to the decomposition of M into eigenspaces for the action of $\Delta \cong (\mathbb{Z}/p)^\times$, something which is a very useful tool in this situation. In fact, in this case we can specify the relationship more precisely.

When $G \cong \mathbb{Z}_p^\times$ with $p \geq 3$, G is the direct product of G_0 and Δ . Suppose $\chi : \Delta \rightarrow \mathbb{Z}_p^\times$ is a primitive character in the dual group, $\widehat{\Delta}$, of Δ . We denote by M^χ the $\Lambda(G)$ -submodule of M picked out by the idempotent of $\mathbb{Z}_p[\Delta]$ corresponding to χ , and upon which Δ acts via χ . Then there is a direct sum decomposition

$$M = \bigoplus_{\chi \in \widehat{\Delta}} M^\chi, \quad (52)$$

and the $\Lambda(\Gamma)$ -rank of M is the sum of the $\Lambda(\Gamma)$ -ranks of the M^χ .

From this we can isolate the following stronger result:

Corollary 2.4 *For $G \cong \mathbb{Z}_p^\times$ and $p \geq 3$*

$$\text{hmrk}_{\Lambda(G)}(M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)) = \text{rank}_{\Lambda(\Gamma)}(M^{\chi^{-1}}), \quad (53)$$

where $\Gamma \times \Delta = G$, with Γ and Δ as above, and χ is any primitive character of Δ .

Remarks

- i) We note that it is possible for a finitely generated $\Lambda(G)$ -module, M , to have $\text{hmrank}_{\Lambda(G)}(M) = 0$ but non-zero $\Lambda(G_0)$ -rank, where G_0 is any pro- p , open subgroup of G . Indeed this is apparent from the proof of Proposition 2.3 in the special case of G isomorphic to \mathbb{Z}_p^\times . Simply take M to be $\Lambda(G)^\chi$ for a non-trivial character $\chi : \Delta \rightarrow \mathbb{Z}_p^\times$. This is isomorphic to $\Lambda(\Gamma)(\chi)$, and hence has $\Lambda(\Gamma)$ -rank equal to 1.
- ii) It is also possible to have $\text{hmrank}_{\Lambda(G)}(M)$ negative. I thank Otmar Venjakob for pointing out the following example. Consider $G \cong \Gamma \rtimes \Delta$, where Γ is taken to be isomorphic to \mathbb{Z}_p and $\Delta \cong (\mathbb{Z}/p)^\times$ acts upon Γ via a non-trivial character, $\omega : \Delta \rightarrow \mathbb{Z}_p^\times$. Assume p is at least 3 and so $\#\Delta$ is prime to p . Then the Hochschild-Serre spectral sequence gives $H_i(G, M) = (H_i(\Gamma, M))_\Delta$ for all i . Take $M \cong \mathbb{Z}_p(\omega)$. Then $H_0(G, M)$ vanishes and $H_1(G, M) = (\text{Hom}(\Gamma, \mathbb{Z}_p(\omega)))_\Delta = \mathbb{Z}_p$. Thus $\text{hmrank}_{\Lambda(G)}(\mathbb{Z}_p(\omega)) = -1$. In this case G is a Poincaré group with dualising module $\mathbb{Q}_p/\mathbb{Z}_p(\omega)$, and I have been unable to find such an example where G has trivial dualising module.

Question If G is a compact, open subgroup with no p -torsion in a connected, reductive algebraic group over \mathbb{Q}_p then G is a Poincaré group whose dualising module is $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial G -action. (This is essentially due to Lazard, [15] with Burt Totaro pointing out how to extend the argument to G not necessarily a pro- p group.) In this case, is it true that the $\text{hmrank}_{\Lambda(G)}(M)$ is always positive for all finitely generated $\Lambda(G)$ -modules?

We would hope that these ideas would generalise, at least to the case $G \cong G_0 \rtimes \Delta$, but there are added complications, especially if Δ is non-Abelian or the representations of Δ do not decompose completely into characters over \mathbb{Z}_p . If we cannot exhibit Δ as a subgroup of G then it is presently not at all clear how to proceed.

2.2 M fails to be finitely generated

A second situation in which the homological definition of rank is of some interest is in considering modules which are not finitely generated. The general situation is rather unclear to us and so, instead of giving a general discussion, we will consider a particular example of one situation which arose in [6]. Let E be an elliptic curve

defined over a number field, F , and with no complex multiplication over the algebraic closure of F . Denote by E_{p^∞} the set of all p^{th} -power torsion points in E , and by F_∞ we mean $F(E_{p^\infty})$, the field of definition of E_{p^∞} . Let G_∞ denote the Galois group of F_∞ over F . It is well known, due to Serre, that the action of G_∞ on E_{p^∞} gives an injection of G_∞ into $\text{GL}_2(\mathbb{Z}_p)$ as an *open* subgroup, under our assumption that E has no complex multiplication. For p at least 5, the group $\text{GL}_2(\mathbb{Z}_p)$ contains no element of order p and thus, from the properties of p -adic Lie groups listed above, G_∞ has p -cohomological dimension equal to 4. We denote by F_n the field $F(E_{p^{n+1}})$ obtained by adjoining the p^{n+1} torsion points on E to F , and by F^{cyc} the cyclotomic \mathbb{Z}_p -extension of F . As is well known, it follows from the Weil pairing that F^{cyc} is contained in F_∞ . We define

$$H = \text{Gal}(F_\infty/F^{\text{cyc}}), \quad H_0 = \text{Gal}(F_\infty/F_0^{\text{cyc}}), \quad \text{and} \quad \Gamma = \text{Gal}(F^{\text{cyc}}/F). \quad (54)$$

Let S be any finite set of primes of F containing all primes dividing p , Archimedean primes of F and primes at which E has bad reduction, and let F^S denote, as before, the maximal extension of F unramified at all primes not contained in S . Then, since F_∞ is an unramified extension of F at primes not in S , F_∞ is contained in F^S . For any, not necessarily finite, extension L of F contained in F_∞ the p^∞ -Selmer group of E over L , which we denote by $\mathcal{S}_p(E/L)$, can be defined by the exactness of the sequence

$$0 \longrightarrow \mathcal{S}_p(E/L) \longrightarrow H^1(F^S/L, E_{p^\infty}) \xrightarrow{\lambda_L} \bigoplus_{\nu \in S} J_\nu(E/L). \quad (55)$$

(More usually local conditions are imposed at all primes, but the choice of the primes in S ensures that this definition is equivalent.) Here we are using the notation

$$J_\nu(E/L) = \varinjlim_{\omega|\nu} \bigoplus_{\omega} H^1(K_\omega, E)(p), \quad (56)$$

where K runs over the finite extensions of F contained in L , ω denotes a prime of K dividing ν and the direct limit is taken with respect to the restriction maps.

Then G_∞ acts continuously on $\mathcal{S}_p(E/F_\infty)$, where the latter module is given the discrete topology, thus this action extends to a continuous action of the Iwasawa algebra, $\Lambda(G_\infty)$. We will in fact mainly consider the Pontrjagin dual

$$\mathcal{C}_p(E/F_\infty) = \widehat{\mathcal{S}_p(E/F_\infty)} \quad (57)$$

which has the structure of a compact $\Lambda(G_\infty)$ -module. Similarly, $\mathcal{C}_p(E/F^{\text{cyc}})$ is a compact $\Lambda(\Gamma)$ -module. It is well known that $\mathcal{C}_p(E/F^{\text{cyc}})$ is a finitely generated

$\Lambda(\Gamma)$ -module. Similarly $\mathcal{C}_p(E/F_\infty)$ is a finitely generated $\Lambda(G_\infty)$ -module (see [6], Theorem 2.9.)

The following conjecture is folklore.

Conjecture 2.5 *Assume $p \geq 3$. If L is any extension of F^{cyc} contained in F^S then*

i) the map λ_L in the exact sequence (55) is a surjection and

ii) the group $H^2(F^S/L, E_{p^\infty}) = 0$.

In fact, as has been pointed out to us by Y. Ochi, it is known that

$$H^2(F^S/F_\infty, E_{p^\infty}) = 0, \quad (58)$$

but this is still only a conjecture in general for arbitrary extensions L of F^{cyc} . We have restricted to $p \geq 3$ because the vanishing of $H^2(F^S/L, E_{p^\infty})$ can fail for $p = 2$.

In the case $L = F^{cyc}$ this is equivalent to a longstanding conjecture on the rank of $\mathcal{C}_p(E/F^{cyc})$ as a $\Lambda(\Gamma)$ -module, originally due to Mazur in the case E has good, ordinary reduction at all primes of F dividing p . In particular, if E has potential good ordinary reduction or potential multiplicative reduction at all primes of F dividing p then for F^{cyc} the following is equivalent to conjecture 2.5.

Conjecture 2.6 $\mathcal{C}_p(E/F^{cyc})$ is $\Lambda(\Gamma)$ -torsion.

It follows from the structure theorem for finitely generated $\Lambda(\Gamma)$ -modules that $\mathcal{C}_p(E/F^{cyc})$ has finite \mathbb{Z}_p -rank in this case (though it will generally not be finitely generated as a \mathbb{Z}_p -module.) We obtain Conjecture 2.5 for general L from the conjecture for F^{cyc} by taking the direct limit over *all* finite extensions of F contained in F^S .

It was noted in [6] that the classical property of being torsion and having μ -invariant zero in the theory of \mathbb{Z}_p -extensions seems to have a natural parallel in the GL_2 theory of the extension F_∞ of F . We recall that for Γ any group isomorphic to \mathbb{Z}_p , if M is a finitely generated $\Lambda(\Gamma)$ -module then M is torsion and has μ -invariant zero if M is actually finitely generated over \mathbb{Z}_p . This is an immediate consequence of the definition of the classical Iwasawa μ -invariant.

We have given above in (33) a definition of μ -invariant for general $\Lambda(G)$ -modules when G is a pro- p , p -adic, Lie group. But certain results discussed in §6 of [6] for Selmer groups of elliptic curves and in [13] for other situations indicate that

another analogue of the μ -invariant being zero in this GL_2 situation is that a finitely generated $\Lambda(G_\infty)$ -module, M , be in fact finitely generated over the subalgebra, $\Lambda(H)$. Unfortunately this is stronger, the two notions are not equivalent.

Lemma 2.7 *Let M be a $\Lambda(G_\infty)$ -module which is finitely generated over $\Lambda(H)$. Then $\mu(M) = 0$ where μ here is as defined in (33), for any choice of pro- p , open subgroup G' of G_∞ .*

Proof The p -torsion submodule $M(p)$ is a finitely generated $\Lambda(H)$ -submodule, and so all of the sub-quotients, $p^i M(p)/p^{i+1}$, have rank 0 over $\Omega(G')$. □

The converse clearly fails. Consider, for example, the module $\Lambda(\Gamma)/p$ with the structure of a $\Lambda(G_\infty)$ -module via the natural projection $\Lambda(G_\infty) \twoheadrightarrow \Lambda(\Gamma)$. This example, however, is rather small. O. Venjakob has given a definition of pseudonull modules for $\Lambda(G)$ -modules which extends the usual definition in the case of $G \cong \mathbb{Z}_p$, [24]. This example is a pseudonull $\Lambda(G_\infty)$ -module.

In a similar vein, the $\Lambda(H_0)$ -rank of M , as defined in (2) above, bears similarities to the λ -invariant in the theory of \mathbb{Z}_p -extensions. (One has to restrict to H_0 for the usual reasons, to ensure that $\Lambda(H_0)$ has no non-trivial torsion.) We would like to propose that a more suitable invariant should be the homological $\Lambda(H)$ -rank defined above. As we shall see, this has computational advantages for modules which are not finitely generated over $\Lambda(H_0)$.

Notation In the sequel we let r denote the number of primes of F^{cyc} at which E has split multiplicative reduction.

Theorem 2.8 *Assume $p \geq 5$ and E does not have potential good supersingular reduction at any prime of F dividing p . Assume conjecture 2.5 for the fields F_∞ and F^{cyc} . Then $\mathcal{C}_p(E/F_\infty)$ has finite homological rank over $\Lambda(H)$. If F contains μ_p , the p^{th} roots of unity, then this is given by*

$$\mathrm{hmrank}_{\Lambda(H)}(\mathcal{C}_p(E/F_\infty)) = \mathrm{rank}_{\mathbb{Z}_p}(\mathcal{C}_p(E/F^{cyc})) + r. \quad (59)$$

Proof We have the following fundamental diagram, with vertical maps given by restriction maps and exact rows as a consequence of our assumption concerning

Conjecture 2.5:

$$\begin{array}{ccccccc}
0 \rightarrow \mathcal{S}_p(E/F_\infty)^H & \rightarrow & H^1(F^S/F_\infty, E_{p^\infty})^H & \xrightarrow{\phi_\infty} & \left(\bigoplus_{\nu \in S} J_\nu(E/F_\infty) \right)^H \\
& \uparrow f & & \uparrow g & & \uparrow h \\
0 \rightarrow \mathcal{S}_p(E/F^{cyc}) & \rightarrow & H^1(F^S/F^{cyc}, E_{p^\infty}) & \xrightarrow{\lambda_{F^{cyc}}} & \bigoplus_{\nu \in S} J_\nu(E/F^{cyc}) & \rightarrow & 0
\end{array} \tag{60}$$

This diagram is analysed in detail in [6], where the following is proved (Lemma 6.7.)

Lemma 2.9 *i) The kernel and cokernel of g are both finite.*

ii) The cokernel of h is finite and the kernel of h has finite \mathbb{Z}_p -corank, at most equal to r .

iii) This is an equality if F contains μ_p .

Since $\mathcal{C}_p(E/F^{cyc})$ has finite \mathbb{Z}_p -rank, by assumption, it follows from the snake lemma that $\mathcal{C}_p(E/F_\infty)_H$ has finite \mathbb{Z}_p -rank, given by

$$\begin{aligned}
\text{rank}_{\mathbb{Z}_p}(\mathcal{C}_p(E/F_\infty)_H) &= \text{rank}_{\mathbb{Z}_p}(\mathcal{C}_p(E/F^{cyc})) + \text{rank}_{\mathbb{Z}_p}(\widehat{\text{Ker}(h)}) \\
&\leq \text{rank}_{\mathbb{Z}_p}(\mathcal{C}_p(E/F^{cyc})) + r,
\end{aligned} \tag{61}$$

with equality if F contains μ_p .

Since we are assuming Conjecture 2.5 for F_∞ , the top row of (60) extends to a long exact sequence in cohomology:

$$\begin{aligned}
0 \rightarrow \text{Coker}(\phi_\infty) \rightarrow H^1(H, \mathcal{S}_p(E/F_\infty)) \rightarrow \dots \\
\rightarrow H^3(H, H^1(F^S/F_\infty, E_{p^\infty})) \rightarrow H^3(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty)) \rightarrow 0
\end{aligned} \tag{62}$$

Since the map $\lambda_{F^{cyc}}$ is surjective and $\text{Coker}(h)$ is finite, it follows from diagram (60) and the snake lemma that the cokernel of ϕ_∞ is finite.

Lemma 2.10 *We assume $p \neq 2$. Then it follows from our assumption of the vanishing of $H^2(F^S/F^{cyc}, E_{p^\infty})$ that the groups $H^i(H, H^1(F^S/F_\infty, E_{p^\infty}))$ are finite for all $i \geq 1$.*

Proof We exploit the fact that it is known that $H^j(F^S/F_\infty, E_{p^\infty}) = 0$ for j at least 2. Then it follows from the Hochschild-Serre spectral sequence that we have the exact sequence

$$H^i(F^S/F^{cyc}, E_{p^\infty}) \longrightarrow H^{i-1}(H, H^1(F^S/F_\infty, E_{p^\infty})) \longrightarrow H^{i+1}(H, E_{p^\infty}), \tag{63}$$

for i at least 2. We have assumed that $H^2(F^S/F^{cyc}, E_{p^\infty})$ vanishes, whilst for $i \geq 3$ the groups $H^i(F^S/F^{cyc}, E_{p^\infty})$ vanish since the p cohomological dimension of $\text{Gal}(F^S/F^{cyc})$ equals 2 under our assumption that p is not equal to 2. The groups $H^i(H, E_{p^\infty})$ are easily seen to be finite for all $i \geq 0$. See, for example, [7].

□

Lemma 2.11 *Assume that $p \geq 5$. Then for all $i \geq 1$ one has*

$$H^i(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty)) = 0. \quad (64)$$

Proof It is proven in [6] that (64) is true with H replaced by G_∞ (Proposition 5.12 for $\nu \nmid p$ and Corollary 5.23 for $\nu \mid p$.) Since $cd_p(\Gamma) = 1$ the Hochschild-Serre spectral sequence gives rise to short exact sequences

$$\begin{aligned} 0 \rightarrow H^j(\Gamma, H^j(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty))) \rightarrow \\ H^{j+1}(G_\infty, \bigoplus_{\nu \in S} J_\nu(E/F_\infty)) \rightarrow H^{j+1}(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty))^\Gamma \rightarrow 0 \end{aligned} \quad (65)$$

Since $H^{j+1}(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty))$ is p -primary and has the discrete topology, and since Γ is isomorphic to \mathbb{Z}_p , the vanishing of $H^{j+1}(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty))^\Gamma$ for $j \geq 0$ implied by (65) ensures that $H^i(H, \bigoplus_{\nu \in S} J_\nu(E/F_\infty))$ vanishes for all $i \geq 1$.

□

Thus we see from the exact sequence (62), Lemmas 2.10 and 2.11 and the finiteness of $\text{Coker}(\phi_\infty)$ that for all $i \geq 1$ the $H^i(H, \mathcal{S}_p(E/F_\infty))$ are finite.

The duality between homology and cohomology described in (5) extends to this situation, even though $\mathcal{C}_p(E/F_\infty)$ is not finitely generated as a $\Lambda(H)$ -module. Indeed, taking Pontrjagin duals passes between the categories of compact $\Lambda(H)$ -modules and discrete $\Lambda(H)$ -modules, without any finite generation assumption. This duality again sends compact, projective modules to discrete, injective modules, and we still have

$$\widehat{M_H} = \widehat{M}^H, \quad (66)$$

for any compact $\Lambda(H)$ -module, without assuming M to be finitely generated. In fact, since finitely generated $\Lambda(G)$ -modules are pseudo-compact as $\Lambda(H)$ -modules, this is explained in greater detail in [3]. Thus we also still have a canonical isomorphism

$$H^i(H, \widehat{M}) \cong H_i(\widehat{H}, \widehat{M}). \quad (67)$$

It follows that $H_i(H, \mathcal{C}_p(E/F_\infty))$ is finite for all $i \geq 1$, and so the homological $\Lambda(H)$ -rank of $\mathcal{C}_p(E/F_\infty)$ is just the \mathbb{Z}_p -rank of $\mathcal{C}_p(E/F_\infty)_H$, given by (61). This completes the proof of Theorem 2.8

□

As a consequence of Theorem 1.1, in the case H is pro- p and $\mathcal{C}_p(E/F_\infty)$ is actually finitely generated over $\Lambda(H)$ we have calculated the true rank, not just the homological rank. Thus we obtain directly the following result (Proposition 6.9) in [6].

Corollary 2.12 *Assume the hypotheses of Theorem 2.8. Assume also that H is a pro- p group and that $\mathcal{C}_p(E/F^{cyc})$ has μ -invariant equal to 0. Then if L is any finite extension of F^{cyc} contained in F_∞*

$$\lambda_{inv}(\mathcal{C}_p(E/L)) = |L : F^{cyc}| \left(\lambda_{inv}(\mathcal{C}_p(E/F^{cyc}) + r) \right) - r_L \quad (68)$$

where r_L denotes the number of primes of L at which E has split multiplicative reduction.

This is clear from the fact that if M is a finitely generated $\Lambda(H)$ -module of rank d and if H' is an open subgroup of H , then M is a finitely generated $\Lambda(H')$ -module of rank $|H : H'| \cdot d$. Corollary 2.12 was originally obtained as a consequence of Hachimori and Matsuno's more general formula concerning the growth of λ -invariants in a p -extension (see [12]). We believe the above proof explains, in this situation, the origin of the terms in their formula.

We conclude this section by illustrating Theorem 2.8 with an example expanding upon one considered in [6].

Example Take E to be the elliptic curve $X_0(11)$ with minimal Weierstraß equation over \mathbb{Q} given by

$$E : y^2 + y = x^3 - x^2 - 10x - 20. \quad (69)$$

Take $p = 5$, a prime at which E has good, ordinary reduction. Let H be the group $\text{Gal}(\mathbb{Q}(E_{5^\infty})/\mathbb{Q}(\mu_{5^\infty}))$. It follows from the work of Lang and Trotter [14] that H is pro- p . Then it is explained in §6 and 7 of [6] that $\mathcal{C}_5(E/\mathbb{Q}(E_{5^\infty}))$ is not finitely generated as a $\Lambda(H)$ -module. However, Corollary 7.10 of [6] states

$$\dim_{K(H)}(\mathcal{C}_5(E/\mathbb{Q}(E_{5^\infty})) \otimes_{\Lambda(H)} K(H)) = 4. \quad (70)$$

Recall that $K(H)$ denotes the skew field of fractions of $\Lambda(H)$. It follows that $\mathcal{C}_5(E/\mathbb{Q}(E_{5^\infty}))$ actually has finite $\Lambda(H)$ -rank.

The only rational prime at which E has non-integral j -invariant is 11. There are 4 primes of $\mathbb{Q}(\mu_{5^\infty})$ over 11. Thus the number r appearing in Theorem 2.8 is 4. It follows from the isogeny invariance of the λ -invariant and the results in [6] that $\mathcal{C}_5(E/\mathbb{Q}(\mu_5^\infty))$ has \mathbb{Z}_5 -rank equal to 0 (although it has non-zero μ -invariant and so is not finite.) Thus Theorem 2.8 in this case states:

$$\text{hmrnk}_{\Lambda(H)}(\mathcal{C}_5(E/\mathbb{Q}(E_{5^\infty}))) = 4, \quad (71)$$

agreeing with the actual $\Lambda(H)$ -rank in (70).

We have unfortunately so far been unable to deduce from Theorem 2.8 the answer to the original question raised in [6], of whether the $\Lambda(H_0)$ -rank is always finite under these conditions. As illustrated by the above, in the very few examples discussed in [6] where we know the $\Lambda(H_0)$ -rank of $\mathcal{C}_p(E/F_\infty)$ it agrees with the $\Lambda(H_0)$ -hmrnk.

3 Variation of the μ -invariant of Elliptic Curves under Isogeny

Recalling the definition (33) of the μ -invariant of modules finitely generated over Iwasawa algebras given in §1.2, we consider how this behaves for Selmer groups of elliptic curves. Perrin-Riou in [19] and the appendix of [18] and Schneider in [20] consider the variation under isogeny of the classical Iwasawa μ -invariant of finite dimensional p -adic representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For simplicity, we will restrict to Selmer groups of elliptic curves, but as in [19], the argument holds more generally if we use Greenberg's definition of Selmer groups for p -adic representations given in [11] and make the appropriate generalisation of the assumption **L** below. We continue with the situation under discussion in §2.2 and maintain all the same notation.

The behaviour of the μ -invariant for Selmer groups of elliptic curves under isogeny has also been considered independently by J.H. Coates and R. Sujatha, who also have an alternative approach to these results.

Assume E_1 and E_2 are non-CM elliptic curves defined over a number field, F , and they are isogenous via $\phi : E_1 \rightarrow E_2$. We suppose $F_\infty = F(E_{1,p^\infty})$ is a pro- p extension of F . Note this field is the same as $F(E_{2,p^\infty})$, see [6] Lemma 7.8. Let A denote the

p -part of the kernel of ϕ . Throughout this section we need to assume a special case of conjecture 2.5.

Assumption L The localisation maps

$$\lambda_{i,F_\infty} : H^1(F^S/F_\infty, E_{i,p^\infty}) \longrightarrow \bigoplus_{\nu \in S} J_\nu(E_i/F_\infty) \quad (72)$$

are surjective for both $i = 1, 2$.

In fact, as explained in [6] Proposition 7.9, if this holds for one of the curves E_i then it automatically holds for the other.

The isogeny, ϕ , induces homomorphisms on the cohomology groups and Selmer groups:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{S}_p(E_2/F_\infty) & \longrightarrow & H^1(F^S/F_\infty, E_{2,p^\infty}) & \xrightarrow{\lambda_{2,F_\infty}} & \bigoplus_{\nu \in S} J_\nu(E_2/F_\infty) & \longrightarrow 0 \\ & \uparrow \phi_1 & & \uparrow \phi_2 & & \uparrow \phi_3 & \\ 0 \longrightarrow & \mathcal{S}_p(E_1/F_\infty) & \longrightarrow & H^1(F^S/F_\infty, E_{1,p^\infty}) & \xrightarrow{\lambda_{1,F_\infty}} & \bigoplus_{\nu \in S} J_\nu(E_1/F_\infty) & \longrightarrow 0 \end{array} \quad (73)$$

We wish to consider the change in the μ -invariant.

Theorem 3.1 *Let $p \geq 5$. Suppose $\phi : E_1 \longrightarrow E_2$ is an isogeny between two elliptic curves which do not admit complex multiplication, as described above. Assume also that the E_i satisfy Assumption L and that F_∞ is a pro- p extension of F . Then, as $\Lambda(G_\infty)$ -modules*

$$\begin{aligned} \mu(\widehat{\text{Coker}(\phi_1)}) - \mu(\widehat{\text{Ker}(\phi_1)}) &= \sum_{\nu \mid \infty} \text{ord}_p(\#(A(F_\nu))) \\ &- \mid F : \mathbb{Q} \mid \text{ord}_p(\#A) - \sum_{\nu \mid p} \text{ord}_p(\mid \# \tilde{A}_\nu \mid_\nu) \end{aligned} \quad (74)$$

where $A = \text{Ker}(\phi(p))$ as above, ν denotes a prime of F , by $A(F_\nu)$ we mean the set of F_ν -valued points in A , and \tilde{A}_ν denotes the image of A under the reduction map at $\nu : E_1 \longrightarrow \tilde{E}_1$.

Proof This follows closely the ideas of Perrin-Riou in the cyclotomic case given in detail for elliptic curves in the appendix to [18]. We proceed similarly to the proof of Theorem 2.8 above, with a close analysis of a diagram, this time (73).

The snake lemma gives

$$\begin{aligned} 0 \longrightarrow \operatorname{Ker}(\phi_1) \longrightarrow \operatorname{Ker}(\phi_2) \longrightarrow \operatorname{Ker}(\phi_3) \longrightarrow \operatorname{Coker}(\phi_1) \\ \longrightarrow \operatorname{Coker}(\phi_2) \longrightarrow \operatorname{Coker}(\phi_3) \longrightarrow 0. \end{aligned} \quad (75)$$

The first thing to remark is that the Pontrjagin dual of the kernels and cokernels of ϕ_2 and ϕ_3 (and hence of ϕ_1 by the snake lemma) are all finitely generated $\Lambda(G_\infty)$ -modules, annihilated by a finite power of p . Note that although Corollary 1.6 is given in terms of group homology, because of (5) it can also be rephrased in terms of cohomology as for Theorem 1.1. For discrete, cofinitely generated $\Lambda(G_\infty)$ -modules, we will use also the notation $\chi(G_\infty, D)$ to denote $\prod_{i \geq 0} \#(H^i(G_\infty, D))^{(-1)^i}$ where the choice between cohomology and homology is dictated by whether the module is discrete or compact, respectively. Since either choice gives zero for finite modules when G_∞ is pro- p , there will be no ambiguity. Thus if M is a finitely generated $\Lambda(G_\infty)$ -module then $\mu(M) = \operatorname{ord}_p(\chi(G_\infty, \widehat{M(p)}))$ also. Then the multiplicativity of $\chi(G_\infty, _)$ along exact sequences gives

$$\frac{\chi(G_\infty, \operatorname{Coker}(\phi_1))}{\chi(G_\infty, \operatorname{Ker}(\phi_1))} = \frac{\chi(G_\infty, \operatorname{Coker}(\phi_2))}{\chi(G_\infty, \operatorname{Ker}(\phi_2))} \times \frac{\chi(G_\infty, \operatorname{Ker}(\phi_3))}{\chi(G_\infty, \operatorname{Coker}(\phi_3))}, \quad (76)$$

where all Euler characteristics are defined. Since $\operatorname{Coker}(\phi_1)$ and $\operatorname{Ker}(\phi_1)$ are both annihilated by a finite power of p , by Corollary 1.7 the number we require for the Theorem is then the logarithm to base p of this.

For the first term, recall that we know $H^i(F^S/F_\infty, E_{i,p^\infty})$ vanishes for $i \geq 2$, by (58) and the assumption that $p \geq 3$. Thus by the long exact sequence in cohomology of which the central vertical arrow is a part, we have $\operatorname{Coker}(\phi_2) = H^2(F^S/F_\infty, A)$. Since isogenies are surjective $H^0(F^S/F_\infty, E_{1,p^\infty}) = E_{1,p^\infty}$, surjects onto $H^0(F^S/F_\infty, E_{2,p^\infty}) = E_{2,p^\infty}$, and so $\operatorname{Ker}(\phi_2) = H^1(F^S/F_\infty, A)$. The full Hochschild-Serre spectral sequence:

$$H^i(G_\infty, H^j(F^S/F_\infty, A)) \Rightarrow H^{i+j}(F^S/F, A) \quad (77)$$

is bounded since G_∞ , $\operatorname{Gal}(F^S/F_\infty)$ and $\operatorname{Gal}(F^S/F)$ all have finite cohomological dimension at p . It follows from this that

$$\frac{\chi(G_\infty, H^0(F^S/F_\infty, A))\chi(G_\infty, H^2(F^S/F_\infty, A))}{\chi(G_\infty, H^1(F^S/F_\infty, A))} = \chi(\operatorname{Gal}(F^S/F), A). \quad (78)$$

Precise details of how Euler Characteristics behave in spectral sequences, and in particular how to obtain this formula, are given in [13]. But since $H^0(F^S/F_\infty, A)$ is just A and in particular is finite, as remarked in §1.2 it has Euler characteristic

equal to one, and thus

$$\frac{\chi(G_\infty, \text{Coker}(\phi_2))}{\chi(G_\infty, \text{Ker}(\phi_2))} = \chi(\text{Gal}(F^S/F), A). \quad (79)$$

Locally, for any prime ν of F in S we define

$$\begin{aligned} C_\nu^i(F_\infty) &= \varinjlim_{\omega|\nu} \prod H^i(F_{n,\omega}, A) \quad \text{if } \nu \nmid p \\ &\quad \varinjlim_{\omega|\nu} \prod H^i(F_{n,\omega}, \tilde{A}_\nu) \quad \text{if } \nu \mid p \end{aligned} \quad (80)$$

where the direct limits are taken with respect to the restriction homomorphisms. Recall that \tilde{A}_ν is the image of A under the reduction map at $\nu : E_1 \rightarrow \tilde{E}_1$. As explained in the local calculations in [6] §5, it follows from the results in [4] that

$$\begin{aligned} J_\nu(E_i/F_\infty) &= \varinjlim_{\omega|\nu} \prod H^1(F_{n,\omega}, E_{i,p^\infty}) \quad \text{if } \nu \nmid p \\ &\quad \varinjlim_{\omega|\nu} \prod H^1(F_{n,\omega}, \tilde{E}_{i,p^\infty}) \quad \text{if } \nu \mid p. \end{aligned} \quad (81)$$

Let X be E_{i,p^∞} if $\nu \nmid p$, whilst if $\nu \mid p$ then X denotes \tilde{E}_{i,p^∞} . Then the groups $\varinjlim_{\omega|\nu} \prod H^2(F_{n,\omega}, X)$ vanish (also explained in [6] §5) and as described above for the global map ϕ_2 , we have the simple descriptions $\text{Ker}(\phi_3) = \bigoplus_{\nu \in S} C_\nu^1(F_\infty)$ and $\text{Coker}(\phi_3) = \bigoplus_{\nu \in S} C_\nu^2(F_\infty)$.

For each prime ν in S we fix a choice of prime F_∞ lying above, which will also be denoted by ν . It follows from Shapiro's Lemma that

$$H^j(G_\infty, C_\nu^i(F_\infty)) = H^j(\Delta_\nu, H^i(F_{\infty,\nu}, X)), \quad (82)$$

where X is as above and Δ_ν is the decomposition group of ν in G_∞ , that is the Galois group of the extension $F_{\infty,\nu}/F_\nu$. Thus

$$\chi(G_\infty, \text{Ker}(\phi_3)) = \prod_{\nu \in S, \nu \nmid p} \chi(\Delta_\nu, H^1(F_{\infty,\nu}, \tilde{A}_\nu)) \times \prod_{\nu \in S, \nu \nmid p} \chi(\Delta_\nu, H^1(F_{\infty,\nu}, A)) \quad (83)$$

and similarly for $\text{Coker}(\phi_3)$. Then, as we saw when considering ϕ_2 , the Hochschild-Serre spectral sequence

$$H^i(\Delta_\nu, H^j(F_{\infty,\nu}, X)) \Rightarrow H^{i+j}(F_\nu, X) \quad (84)$$

gives

$$\frac{\chi(G_\infty, \text{Ker}(\phi_3))}{\chi(G_\infty, \text{Coker}(\phi_3))} = \frac{1}{\prod_{\nu \in S, \nu \nmid p} \chi(\text{Gal}(\overline{F}_\nu/F_\nu), \tilde{A}_\nu) \times \prod_{\nu \in S, \nu \nmid p} \chi(\text{Gal}(\overline{F}_\nu/F_\nu), A)}. \quad (85)$$

Since A is a p -group, it is known, due to Tate [16] Lemma I.2.9, that the Euler Characteristic corresponding to primes in S which do not lie above p equals one. Then substituting (79) and (85) into (76) and taking the logarithm to base p gives

$$\begin{aligned} \mu(\widehat{\text{Coker}(\phi_1)}) - \mu(\widehat{\text{Ker}(\phi_1)}) &= \text{ord}_p(\chi(\text{Gal}(F^S/F), A)) \\ &- \sum_{\nu \in S, \nu|p} \text{ord}_p(\chi(\text{Gal}(\overline{F}_\nu/F_\nu), \tilde{A}_\nu)). \end{aligned} \quad (86)$$

We recall here Tate's formulae for calculating such Euler characteristics. If M is a finite $\text{Gal}(F^S/F)$ -module such that all primes ν of F dividing the order of M are contained in S , then (see [16] Theorem I.5.1)

$$\chi(\text{Gal}(F^S/F), M) = \prod_{\nu \text{ Arch}} \frac{\#H^0(\overline{F}_\nu/F_\nu, M)}{|\#M|_\nu}. \quad (87)$$

If ν is a non-Archimedean prime of F and M is a finite $\text{Gal}(\overline{F}_\nu/F_\nu)$ -module, then (see [16] Theorem I.2.8)

$$\chi(\text{Gal}(\overline{F}_\nu/F_\nu), M) = |\#M|_\nu. \quad (88)$$

Substituting these into (86) gives precisely the formula (74) in the Theorem. \square

Corollary 3.2 *Maintaining the assumptions of Theorem 3.1, suppose also that the Pontrjagin duals of the p^∞ -Selmer groups, $\mathcal{C}_p(E_i/F_\infty)$, are both $\Lambda(G_\infty)$ -torsion. Let L be any finite extension of F contained in F_∞ . Let G_L denote $\text{Gal}(F_\infty/L)$, an open subgroup of G_∞ . If M is a finitely generated $\Lambda(G_\infty)$ -module, we use the notation μ_L to denote the μ -invariant of M , as defined by (33), as a $\Lambda(G_L)$ -module. Then*

$$\begin{aligned} \mu_L(\mathcal{C}_p(E_2/F_\infty)) - \mu_L(\mathcal{C}_p(E_1/F_\infty)) &= \sum_{\omega|\infty} \text{ord}_p(\#A(L_\omega)) \\ &- |L:\mathbb{Q}| \text{ord}_p(\#A) - \sum_{\omega|p} \text{ord}_p(|\#\tilde{A}_\omega|_\omega) \end{aligned} \quad (89)$$

where ω denotes a prime of L , and other terms are then as defined in Theorem 3.1.

Remark We note that, as explained in [6] Proposition 7.9, the $\Lambda(G_\infty)$ -rank of $\mathcal{C}_p(E/F_\infty)$ is isogeny invariant.

Proof This follows immediately from Theorem 3.1. Since we are now assuming the $\mathcal{C}_p(E_i/F_\infty)$ are $\Lambda(G_\infty)$ -torsion (and thus also $\Lambda(G_L)$ -torsion for all finite extensions, L of F) we know from Proposition 1.8 that

$$\mu_L(\widehat{\text{Coker}(\phi_1)}) - \mu_L(\mathcal{C}_p(E_2/F_\infty)) + \mu_L(\mathcal{C}_p(E_1/F_\infty)) - \mu_L(\widehat{\text{Ker}(\phi_1)}) = 0. \quad (90)$$

Replacing F by L and G_∞ by G_L in Theorem 3.1, and substituting the formula thus resulting from (74) into (90), gives the Corollary.

□

Example We consider the isogeny class of non-CM elliptic curves of conductor 11 containing the curve $X_0(11)$ considered above, (69). Take E_1 to be the curve $X_1(11)$, given by a minimal Weierstraß equation $y^2 + y = x^3 - x^2$, and let E_2 be $X_0(11)$. They are related via the isogeny $\phi : E_1 \rightarrow E_2$ with kernel $A \cong \mathbb{Z}/5$. As remarked in the example at the end of §2, it follows from Lang and Trotter's determination of G_∞ in this case that $\text{Gal}(\mathbb{Q}(E_{i,5^\infty})/\mathbb{Q}(\mu_5))$ is pro-5, [14]. Letting L denote any finite extension of $\mathbb{Q}(\mu_5)$ contained in $\mathbb{Q}(E_{i,5^\infty})$ and so G_L is the group $\text{Gal}(E_{i,5^\infty}/L)$, it is shown in [6], §6 and 7, that both $\mathcal{C}_5(E_i/F_\infty)$ are $\Lambda(G_L)$ -torsion, and thus by Proposition 3.4 of [6] Assumption **L** holds. Then Corollary 3.2 applies. Since L contains $\mathbb{Q}(\mu_5)$, all Archimedean places of L are complex. Then

$$\begin{aligned} \sum_{\omega|\infty} \text{ord}_5(\#A(L_\omega)) &= \frac{1}{2} |L : \mathbb{Q}|, \\ |L : \mathbb{Q}| \text{ord}_5(\#A) &= |L : \mathbb{Q}|. \end{aligned} \tag{91}$$

Turning now to local considerations, the kernel of the isogeny $\phi : E_1 \rightarrow E_2$ injects into $\tilde{E}_{1,5^\infty}$ and so $\tilde{A}_\omega \cong \mathbb{Z}/5$ also, for each prime ω of L above 5. Thus

$$\sum_{\omega|5} \text{ord}_5(|\# \tilde{A}_\omega|_\omega) = -|L : \mathbb{Q}|. \tag{92}$$

It is also explained in [6], Corollary 7.3, that $\mathcal{C}_5(E_1/F_\infty)$ is a finitely generated $\Lambda(H)$ -module for $H = \text{Gal}(F_\infty/L^{cyc})$. Thus, by Lemma 2.7, $\mu_L(\mathcal{C}_5(E_1/F_\infty)) = 0$. We are using here again the notation μ_L defined in Corollary 3.2. From this, the calculations (91) and (92) above and Corollary 3.2 we see that as a $\Lambda(G_L)$ -module

$$\mu_L(\mathcal{C}_5(E_2/\mathbb{Q}(E_{2,5^\infty}))) = \frac{1}{2} |L : \mathbb{Q}|. \tag{93}$$

Similar calculations can be made also for the third curve in the isogeny class.

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